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# A Study of Multifractal Behavior Observed in Nucleus-Nucleus Collisions at Relativistic Energies

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**Abstract**

*An analysis of a data has been carried out to study multifractality in 28Si-AgBr collisions at* $14.5A\ GeV/c$. *We have studied this for different orders of the moment, i.e. from* $q = -6\ to\ 6$. *This study was conducted on an event-by-event basis. Variations in the generalised fractal dimension,* $Dq$ *, and exponent parameter,* $\tau q$ *, with the q values exhibit the presence of a self-similar nature during the multi-particle production process. The experimental results have been compared with data produced from Heavy Ion Jet Interaction Generator (HIJING) and Ultra-relativistic Quantum Molecular Dynamics (UrQMD) models.*

**Keywords:** Multifractality, Generalised fractal dimension, Exponent parameter, 28Si–AgBr collisions; Heavy-ion interactions; HIJING; UrQMD

## 1. Introduction

A systematic analysis of nucleus-nucleus (AA) collisions became easy to understand with the availability of a beam of heavy ions. Numerous attempts have been made to construct colliders like the Large Hadron Collider (LHC) at CERN and the Relativistic Heavy Ion Collider (RHIC) at BNL to understand the underlying dynamics of nuclear collisions at higher centre-of-mass energies. Quark–gluon plasma (QGP) was supposed to persisted in the early cosmos [3]. It might be believed that the innermost region of a neutron star still possesses QGP [2]. The condition suitable to produce QGP can be achieved in the laboratory by colliding two heavy nuclei at relativistic energies and thus creating a little bang. When nucleons collide, they deposit a high concentration of energy within a confined space, which results in an extremely high energy density (of the order of GeV/fm3) for a short span of time. Nuclear collisions at LHC and RHIC energies are currently providing the expected extreme conditions of energy densities and temperature and leading to the deconfined state of quarks and gluons for a short time. The study of the indirect signals of such a state is sufficient to unravel the secrets that lead to the formation of this interesting state of matter, i.e., QGP.

A thorough study of various aspects of particles produced in AA collisions at different relativistic energies involving different projectiles, for example, pseudorapidity distribution, multiplicity distribution, mean multiplicity, and its dispersion, scaling of multiplicities, correlation, fluctuation, and clusterisation in terms of various types of moments (Gq, Fq, and

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Tq) including their dependence on the projectile energy, projectile/target size, rapidity windows, etc., may provide some vital informations regarding the process of particle production in these interactions.

The final-state charged particles resulting from high-energy collisions show rapid fluctuations, with both dips and spikes. There may be two different reasons for such fluctuations: (i) statistical fluctuations [4] because of the limited number of final state charged particles produced in an interaction, and (ii) some dynamical fluctuations which can not be measured directly in an experiment. The impact of statistical noise can be significantly diminished when the density function is averaged over the entire number of events. However, the dynamical component of these fluctuations is also averaged out at the same time, which leads to a smooth distribution of the final state particles. Furthermore, an investigation into the dynamic source of localised fluctuations in JACEE events [1, 5] gave further impetus to these studies. The scaling of factorial moments refers to a non-trivial relationship between these moments and the size of the phase space bins. This behaviour is termed in- termittency and introduced by Bialas and Peschanski [6,7] in particle physics. They applied the concept of intermittency to the JACEE events induced by ultra-high-energy cosmic ray nuclei [1, 5]. Such anomalous scaling has been observed in numerous experiments [8–11].

The term "intermittency" is closely related to the concept of fractal geometry in the context of the underlying physical processes [15]. The physical significance of intermittency is that it refers to the irregular and discontinuous nature of certain phenomena, where events or fluctuations occur sporadically rather than continuously. In order to mathematically describe a system that exhibits intrinsic irregularities at all scales, fractal geometry is utilised. Upon magnifying a small portion of the fractal structure, it exhibits the complexity of the entire system. Thus, it is interesting to employ a formalism that can describe systems with the local characteristics of self-similarity. A number of methods have been suggested to study the fractal structure of multiparticle production in high-energy collisions [10, 11, 13]. We have used the method of multifractal moment in our analysis. We have investigated the multifractal parameters for Monte Carlo (MC) generated events. The MC models used are HIJING and UrQMD. The simulated data is then compared with the results obtained from the experimental data.

## 2. DETAILS OF THE DATA

**Experimental:** The data were collected using a nuclear emulsion stack exposed to 14.5 A GeV/c 28Si nuclei. The size of the pellicles was $16.0 \times 10.0 \times 0.06$ cm$^3$. For the search for an interaction/event, the line scanning method was implemented. To ensure any probability of mixing of primary tracks with secondary interactions, the primary events were traced returning to the corner of the pellicles. When a relativistic particle interacts with the nuclei of emulsion, it results in the production of a high multiplicity of particles. The produced particles are categorised based on their specific ionisation, $g*(= g/g0)$, where g and g0 are respectively the ionisations of the considered track and the ionisations of the primary charged particle. The produced particles appear in the form of tracks of emul- sion with with $g* > 10$, $1.4 \leq g* \leq 10$ and $g* < 1.4$ are termed as black, grey and shower tracks, respectively. The number of showers, grey and black tracks in an event are denoted by ns, ng and nb. The number of black and grey to tracks taken together are known and heavy, i.e nh = nb + ng. The events with light emission targets, i.e. carbon (C), nitrogen (N) and oxygen (O) are the events having nh⟨8. The events with nh ≥ 8, are due to heavy emulsion nuclei, i.e. silver (Ag) and Bromine (Br). Based on this criterion, we have randomly selected 555 clean events with nh ≥ 8 on which the present study is based. The rest of the details may be found in one of our earlier publications [18].

**UrQMD:** The UrQMD model, which is a transport model employed to simulate and study various aspects of high-energy heavy-ion interactions, particularly at high temperatures and energies. The energy range to use this model is from 0.5 GeV per nucleon to 200 GeV per nucleon. It is a fully integrated MC simulation package for proton-proton (pp), proton-nucleus (pA), and AA inter- actions. The model can be used to explore the properties of nuclear matter, including its response to high-energy collisions, compression, and expansion.

UrQMD provides a comprehensive framework for addressing many research goals, allowing physicists to gain insights into the fundamental properties of matter in highly challenging conditions. It is particularly valuable in the field of heavy-ion physics, where the collision of heavy nuclei at high energies can produce unique phenomenon for study.

**HIJING:** HIJING is a MC computer program used to study high-energy collisions. This model is designed to simulate and analyse the production of jets and particles in these collisions, particularly in the context of high-energy heavy-ion collisions. The HIJING model is based on Quantum Chromodynamics (QCD), which is the theory of the strong nuclear force. It also includes soft excitation, multiple minijet production, nuclear shadowing of partons, and final state particles. The key factor of the HIJING model is its view of AA collisions as a superposition of multiple nucleon-nucleon collisions. This model is commonly used in high-energy heavy-ion collisions to provide a baseline prediction to compare the MC-based simulation results with the experimental data.

## 3. THE MATHEMATICAL FORMALISM:

The multifractal moments, Gq, play a crucial role in the study of particle production in nuclear reactions, particularly in the context of understanding the cascading mechanism. These moments are expressed as functions of the pseudorapidity bin size and the order of the moments, q. The calculation of these moments is instrumental in investigating the multifractal properties associated with the multiparticle production process, shedding light on the underlying dynamics of hadronisation in AA collisions.

The framework for this multifractal analysis was introduced by R. C. Hwa [8]. This formalism provides a systematic approach for studying the fractal characteristics of particle production.

The primary goal of this technique is to assess the validity of the fundamental scaling properties inherent to multifractal theories when they are applied to particle production. This analysis seeks to elucidate the dynamic processes responsible for the hadronisation phenomenon in AA collisions. The concept of cascading, as observed in this context, is akin to self-similar processes reminiscent of geometrical objects, such as fractals [14].

In essence, the calculation and investigation of multi-fractal moments within different pseudorapidity bin sizes and moment orders serve as a vital tool for understanding the intricate and fractal-like nature of particle production in nuclear collisions. This approach not only furthers our comprehension of the dynamics involved but also aids in the validation of scaling laws inherent in multifractal theories. It is through these multifractal moments and the cascading mechanism that we gain deeper insights into the fascinating world of particle physics and the phenomenon occurring within the QGP [14].

The pseudorapidity, η, which is defined as $\eta = -ln\,(tan\,\theta/2)$, where θ is the angle between a secondary particle and the beam direction [12]. To examine the η−dependence of $G_q$, in a given η−range (Δη = ηmax − ηmin) is divided into M bins of width δη = Δη/M . The $G_q$ is defined as:

$$G_q = \sum_{j=1}^{M} p_j^q$$

Where,

$$p_j^q = \frac{n_j}{n}$$

Such that

$$n = n_1 + n_2 + n_3 + \cdots + n_M$$

where M represents the total number of non-empty bins. q is a real number and may have either negative or positive values. The ⟨Gq⟩ is calculated by averaging the Gq over the entire ensemble

$$\langle G_q \rangle = \frac{1}{N} \sum_{1}^{M} G_q$$

where N represents the total number of events.

If the multiparticle production has undergone the process of self-similarity, then the $G_q$ moments can be expressed in the form of a power law

$$\langle G_q \rangle = (\delta\eta)^{\tau_q}$$

For $\delta\eta \to 0$

Where $\tau_q$ represents the mass exponent. $\tau_q$ are related as

$$\tau_q = \lim_{\delta\eta \to 0} \frac{\Delta ln\langle G_q \rangle}{\Delta ln\delta\eta}$$

The multifractal spectrum $f(\alpha_q)$ and $\tau_q$ are related as

$$\alpha_q = d\frac{d\tau_q}{dq}$$

The spectral function is obtained by the by Legendre transformations:

$$f(\alpha_q) = q\alpha_q - \tau_q$$

There are some properties of $f(\alpha_q)$ that defines the multifractal behaviour of particles in heavy-ion collisions:

$$\frac{df(\alpha_q)}{dq} = q \text{ and } \frac{d^2 f(\alpha_q)}{dq^2} < 0$$

The curve of $f(\alpha_q)$ has the following properties for multifractals:

- downward concaveness of $f(\alpha_q)$ is.
- Maximum of $f(\alpha_q)$, for $q = 0$
- $f(\alpha_0) > f(\alpha_q)$ for $q \neq 0$

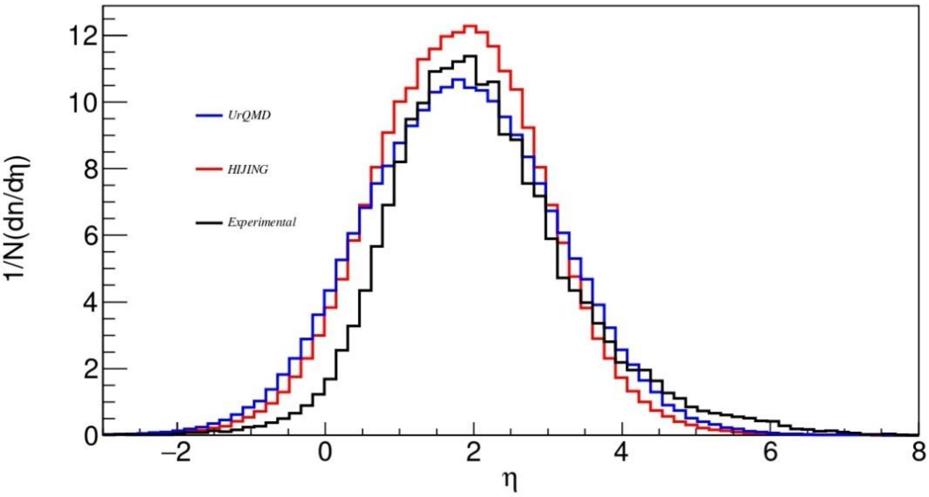
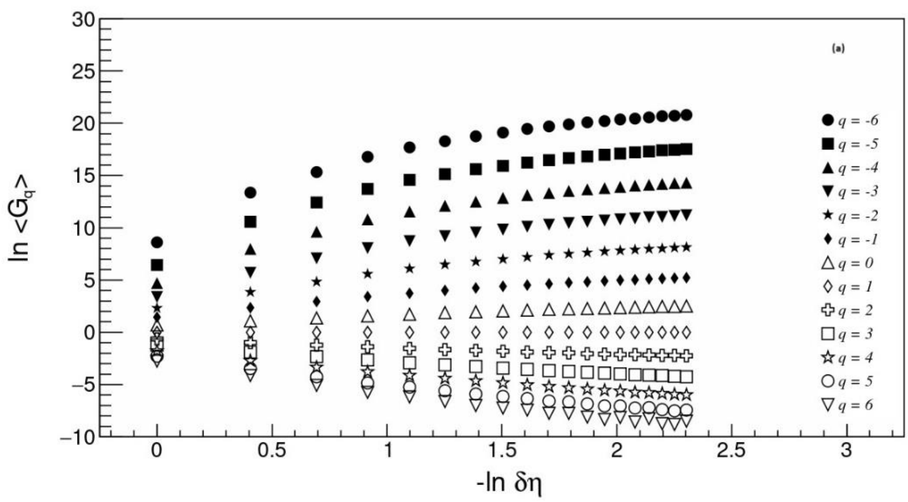



**Fig. 1:** Pseudorapidity distribution for UrQMD, HIJING, and Experimental data.

The concept of the function $f(\alpha_q)$ relates to the measurement of fluctuations within a system using a metric based on the width of a function. In this context, the width of the function serves as an important parameter for quantifying the extent of fluctuations. The narrower width of the function indicates relatively small fluctuations, while a wider width suggests larger and more significant volatility. Additionally, the value of $f(\alpha_q)$ being less than one provides us with an insightful indicator of the number of empty bins within a dataset or system. When $f(\alpha_q)$ is below one, it implies that there is a noticeable presence of empty bins or unoccupied data points.



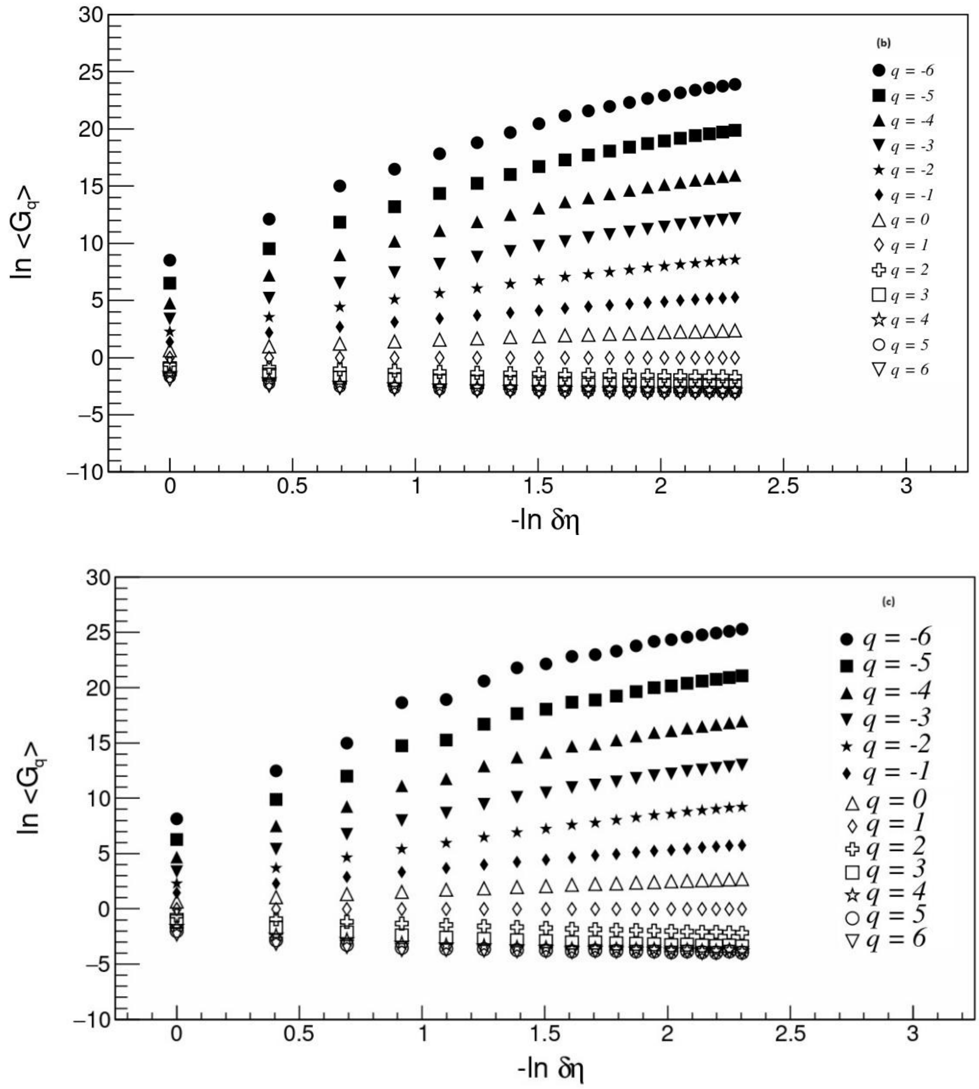



**Fig. 2:** Variations of ln⟨Gq ⟩ with -lnδη for (a) UrQMD, (b) HIJING and (c) Experimental data.

These two metrics, the width of $f(\alpha_q)$ and the value of $f(\alpha_q)$< 1, are essential tools for characterising and comprehending the behaviour of systems and datasets, offering valuable insights into the size of fluctuations and the prevalence of empty bins within a given context.

If there is no absolute fluctuation, then $f(\alpha_q) = \alpha_q = 1$ for all q values. The width of $f(\alpha_q)$ is a measure of the size of the fluctuations, and the value $f(\alpha_0)$<1 is a measure of the number of empty bins [10]. The generalised fractal dimension, denoted as $D_q$, may be defined as:

$$D_q = \frac{\tau_q}{q-1}$$

If we put $q = 0, 1, 2$ in equations $f(\alpha_q)$ and $D_q$, we get

(1) $D_0 = f(\alpha_0)$; Capacity dimension

(2) $D_1 = f(\alpha_1) = \alpha_1$; Entropy dimension

**Table 1:** Variation of $\eta_{mean}$ for experimental, UrQMD and HIJING data.

| **Data Type** | $\langle \boldsymbol{\eta} \rangle$ |
|---|---|
| UrQMD | $1.82651 \pm 0.0154231$ |
| HIJING | $2.11057 \pm 0.0178529$ |
| Experimental | $1.79965 \pm 0.0759811$ |

## 4. RESULT AND DISCUSSION

Here we studied 28Si-AgBr interactions based on 555 events, as mentioned in the details of the data. For comparison, we have generated 14000 events using UrQMD and HIJING event generators. A kinematical cut has been applied to $\eta$ values in the range $\langle \eta \rangle - 1 \leq \eta \leq \langle \eta \rangle + 1$. The values of mean pseudo-rapidity, $\eta$, for experimental and generated events are given in Table 1. A power law behaviour is observed for all the data sets. It may be noted that $ln\langle G_q \rangle$ shows an almost linear behaviour for all values of $q$. This represents a self-similar behaviour in the multiparticle formation process, i.e. a scale independence from $\eta$-width. We may further notice that for negative $q$ values, $ln\langle G_q \rangle$ shows a sharp increase with decreasing bin width. However, for positive $q$ values, $ln\langle G_q \rangle$ decreases slowly with the decreasing bin width. For all $ln \langle G_q\rangle$ values regarding the comparison of UrQMD data with experimental data, we can say that $ln\langle G_q \rangle$ is in good qualitative agreement for negative $q$ values, whereas for positive $q$ values, even if qualitative agreement is absent.

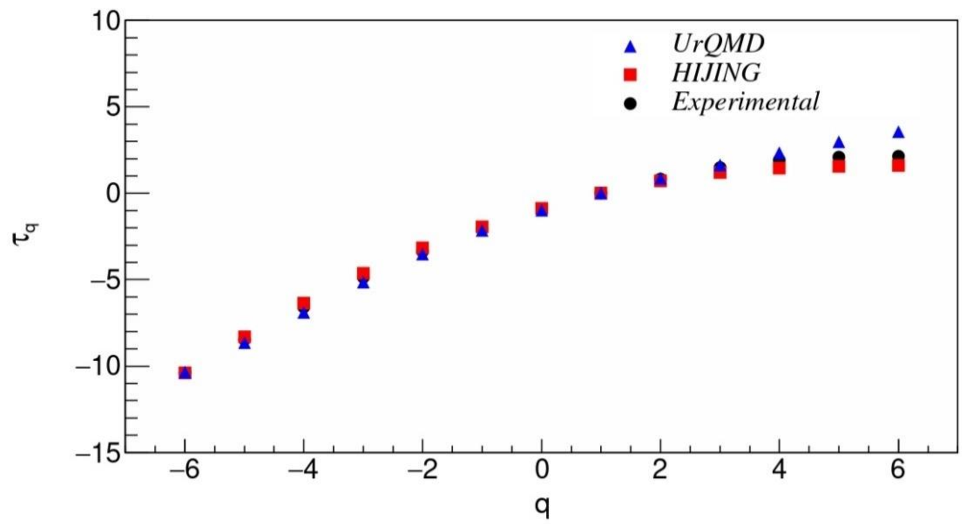



**Fig. 3:** Variation of τq versus q for 28Si-AgBr collisions at 14.5A GeV/c.

In Fig. 3, we have plotted $\tau_q$versus q. It can be seen from Fig. 3 that $\tau_q$ shows a linear dependence with the negative values of q, while it shows saturation behaviour for positive q-values. This may be due to the disparity in behaviour of that $ln\langle G_q \rangle$ for positive and negative values of $q$, as observed in Fig. 2. It can also be noticed that for negative q values, the $\tau_q$ for UrQMD and HIJING agrees well with the experimental data. However, for higher values of $q$, UrQMD is showing slightly higher values than the experimental data. However, for positive values of $q$, UrQMD data show slightly higher values of $\tau_q$ in comparison to experimental data.

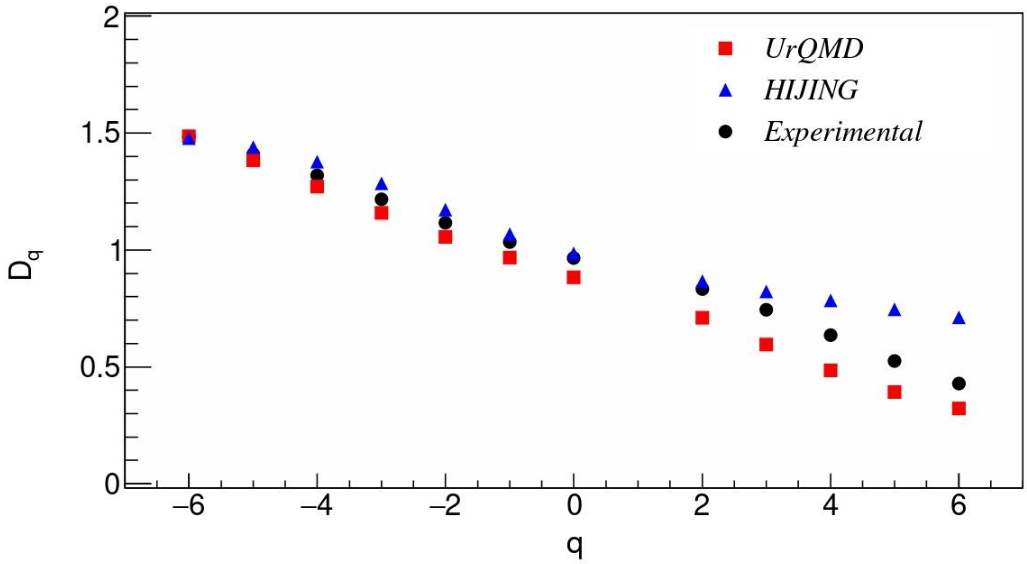



**Fig. 4:** Plot of Dq against q for 28Si-AgBr collisions at 14.5A GeV/c.

Figure 4. represents the graph for $D_q$ as a function of $q$. It can be observed that the $D_q$ shows a negative correlation with increasing $q$ values. The decreasing nature of $D_q$ exhibits the underlying multifractal behaviour while particles are being produced [21]. On comparison with experimental data, a similar qualitative trend of $D_q$ spectrum can be observed for UrQMD and HIJING data. The UrQMD and experimental data show a slow decrease at negative values of $q$ and a sharp decreasing trend at positive $q$ values. However, the measurement of $D_q$ for UrQMD events is smaller in comparison to the experimental data across all $q$ values.

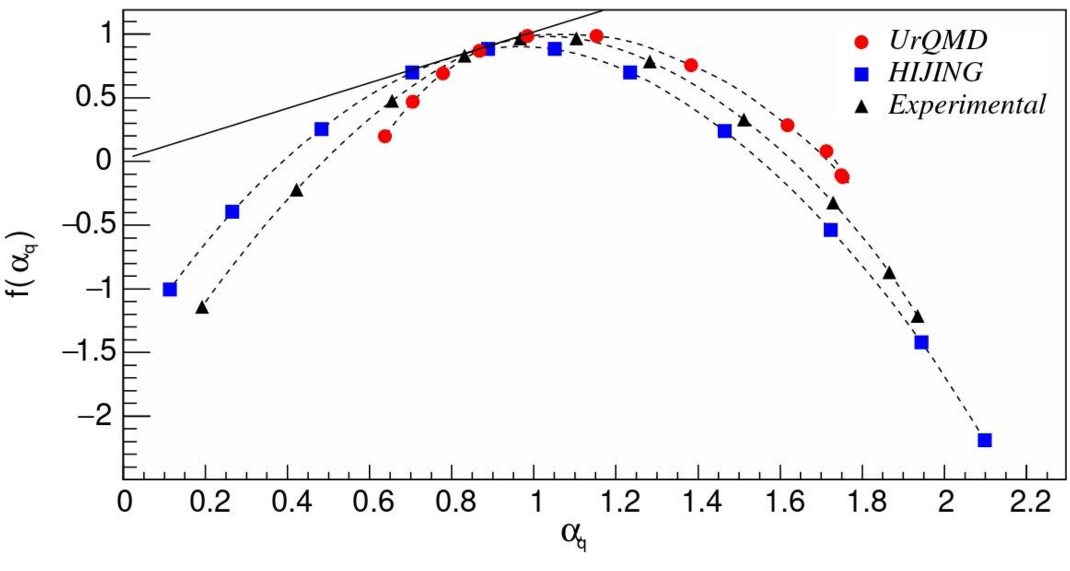



**Fig. 5:** shows the variation of $f(\alpha_q)$ against $f(\alpha_q)$ versus αq for 28Si-AgBr collisions at 14.5A GeV/c.

From this figure, we noticed that there is a downward concave. The downward concavity of the $f(\alpha_q)$ spectrum reflects the characteristic of multifractal behaviour in the multiparticle production process during heavy-ion collisions.

This plot is wide enough to indicate the presence of self-similarity in the final state of multiparticle mass production. Qualitatively, all three data sets follow a similar pattern and show the presence of multifractality. But, if we observe quantitatively, the broadness of UrQMD data is less, indicating weak self-similarity as compared to the experimental data. The figure we can find the information about the fractal dimension $D_0 = f(\alpha_q)$. The solid line in the spectrum depicts a mutual tangent at an angle of $45^0$ at $\alpha_1 = f(\alpha_1)$.

## 5. Conclusions

We have analysed 28Si-AgBr interactions by considering the multifractal moments method.

The experimental data have been compared with the MC models i.e., HIJING and UrQMD. A power-law behaviour is found to be present in all three data sets. The HIJING data agree fairly well with the experimental data, while UrQMD exhibits the qualitative agreement only for negative q values. The power-law behaviour signifies self-similarity in the multi-particle production. The mass exponent, τq, exhibits a linear increase at the negative values of q, while it saturates for the positive values of q. The generalised fractal dimension, Dq, is observed to decrease with the increasing q. The decreasing nature of Dq exhibits the underlying multifractal behaviour through the particle production. The variation of f (αq) with αq shows a downward concave, which again corroborates the presence of multifractal behaviour. All three data sets follow a similar pattern, but the broadness of UrQMD data is lesser as compared to other two data. Thus, we can say that overall, HIJING events are in good agreement with the experimental data as compared to UrQMD.

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